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# An axiomatic approach to the measurement of envy*

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**Abstract**

We characterize a class of envy measures. There are three key axioms. Decomposability requires that overall envy is the sum of the envy within and between subgroups. The other two axioms deal with the two-individual setting and specify how the envy measure should react to simple changes in the individuals' commodity bundles. The characterized class measures the envy of one individual to another by the relative utility difference (using the envious' utility function) between the bundle of the envied and the bundle of the envious. The particular utility representation to be used is fixed by the axioms. The class measures overall envy by the sum of these (transformed) relative utility differences. We discuss our results in the light of previous contributions to envy measurement and multidimensional inequality measurement.

**Keywords.** Envy · Inequality measurement · Decomposability

**JEL classification.** D63

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## 1 Introduction

An allocation is envy-free if no individual prefers another individual's commodity bundle to his own.[1] Envy-freeness is a crude criterion of distributive justice. It distinguishes only two classes of allocations, those that are envy-free and those that are not.

There are good reasons to consider envy measures that provide more discriminatory envy rankings of allocations. Envy-freeness generalizes the idea of equality to the setting of ordinal non-comparable preferences.[2] The study of envy measures is therefore a natural extension of the theory of inequality measurement. Further, allocations that are both envy-free and Pareto efficient are not guaranteed to exist in non-transferable-commodities or production settings.[3] Hence, allocations in the Pareto efficient subset that minimize an envy measure constitute interesting compromises.

We introduce a new class of envy measures. Throughout, we discuss the connections with envy measures proposed by Feldman and Kirman (1974), Chaudhuri (1986), Diamantaras and Thomson (1989) and Fleurbaey (2008). Whereas these previous envy measures were proposed on the basis of their direct appeal, we use an axiomatic approach in order to make intuitions explicit. We develop our class in two steps.

First, we examine the consequences of imposing decomposability. This axiom requires that, for each partitioning of the population in two subgroups, envy in the total population can be written as the sum of the envy within subgroups and the envy between subgroups. In combination with a standard normalization axiom, decomposability implies that envy is measured by $\sum_i \sum_j E_{ij}$, where $E_{ij}$ represents the envy of individual $i$ towards individual $j$ and depends only on the bundles of $i$ and $j$ and the preferences of $i$.

[1] The seminal references are Tinbergen (1946), Foley (1967), Kolm (1971) and Varian (1974). See Arnsperger (1994) and Thomson (2010) for surveys.

[2] See Arnsperger (1994, pp. 157-158) and Fleurbaey (2008, pp. 22-24). For example, if there is only one commodity, say income, then the equal income distribution is the unique envy-free allocation. See Temkin (1986, 1993) and Ebert and Cowell (2004) for approaches to income inequality measurement that explicitly refer to envy-freeness.

[3] See Tadenuma (2002) for a discussion of the clash between Pareto efficiency and envy-freeness in a formal setting similar to ours.

The value of $E_{ij}$ is zero if $i$ does not envy $j$, and positive otherwise.

Second, we formulate two axioms, betweenness and proportionality, that deal with envy comparisons in the simple two-individual setting. Assume that the two individuals are $i$ and $j$, and that $i$ envies $j$ but not vice versa. Betweenness demands that envy decreases if $i$'s bundle improves or if $j$'s bundle worsens according to $i$'s preferences. Proportionality requires that, for cases where the bundles of individuals $i$ and $j$ are proportional to each other, envy is smaller if the radial distance between the bundles is smaller. We show that betweenness and proportionality are incompatible. We weaken the latter axiom to $r$-proportionality, which applies the idea of proportionality only if the bundles of $i$ and $j$ are proportional to a predetermined reference bundle $r$. Given anonymity, betweenness and $r$-proportionality imply that $E_{ij}$ is an increasing function of the ratio $u_i(x_j)/u_i(x_i)$, where $x_i$ and $x_j$ are the bundles of individuals $i$ and $j$ and $u_i$ is a utility representation of $i$'s preferences. The utility representation $u_i$ is not arbitrary, but rather is determined by the axioms and depends on the chosen reference bundle $r$.

The next section introduces notation and provides an overview of the envy measures proposed in the literature. Sections 3 and 4 tackle the two steps discussed above. Section 5 combines the two steps into a single class of envy measures and discusses its properties. Section 6 concludes with a brief discussion of the connection with inequality measurement.

## 2 Preliminaries

The set of individuals is $\mathcal{N}$, a finite subset of the set of positive integers. There are $m$ commodities. The set of commodity bundles is $X = \mathbb{R}^m_{++}$. Each individual $i$ in $\mathcal{N}$ has a preference relation $R_i$, a complete and transitive binary relation on $X$. The strict preference and indifference relations corresponding to $R_i$ are denoted by $P_i$ and $I_i$, respectively. Let $\bar{R}$ be a preference relation such that $x\bar{R}y$ for all bundles $x$ and $y$ in $X$. The preference relation $\bar{R}$ is indifferent between all bundles in $X$ and will play the role of a dummy preference relation. Let $\mathcal{R}$ be the union of $\{\bar{R}\}$ and the set of all continuous and strictly monotonic preference relations. Each individual $i$ in $\mathcal{N}$ has a preference relation in $\mathcal{R}$. For a set of individuals $N \subseteq \mathcal{N}$, we let $x_N = (x_i)_{i \in N}$

and $R_N = (R_i)_{i \in N}$. We refer to $(x_N, R_N)$ as a social state. We do not distinguish between two social states that differ only with respect to the order in which the individuals are listed (e.g., $(x_i, x_j, R_i, R_j)$ and $(x_j, x_i, R_j, R_i)$ are treated as the same social state). The set $S$ collects all social states for all finite population sizes. That is, $S = \bigcup_{N \subset \mathcal{N}} X^{|N|} \times \mathcal{R}^{|N|}$.

Consider a social state $s = (x_N, R_N)$. Individual $i$ is said to envy individual $j$ if $x_j P_i x_i$. The social state $s$ is said to be envy-free if $x_i R_i x_j$ for all individuals $i$ and $j$ in $N$. We use an envy measure to rank all social states in $S$ on the basis of envy. An envy measure is a function $E : S \to \mathbb{R}$ that associates with each social state $s$ in $S$ a level of envy $E(s)$.

We define two basic axioms. More axioms will be introduced in the subsequent sections. Normalization requires the envy measure to attain the value of zero in envy-free social states and positive values in other social states.

**Normalization.** For each social state $(x_N, R_N)$ in $S$, we have $E(x_N, R_N) \geq 0$ with equality holding if and only if $x_i R_i x_j$ for all individuals $i$ and $j$ in $N$.

Anonymity demands that two social states featuring identical bundle-preference pairs have the same level of envy. These two states may distribute these identical preference-bundle pairs differently over the same population or over altogether different populations (of the same size). For a bijection $\pi : N \to M$, and a social state $(x_N, R_N)$, we write $\pi(x_N)$ for $(x_{\pi(i)})_{i \in N}$ and $\pi(R_N)$ for $(R_{\pi(i)})_{i \in N}$.

**Anonymity.** For each social state $(x_N, R_N)$ in $S$ and each bijection $\pi : N \to M$, we have $E(x_N, R_N) = E(\pi(x_N), \pi(R_N))$.

It will be useful to single out two social states induced by a social state $s = (x_N, R_N)$ in $S$. First, let $s_i$ denote the social state in which the preference relation of each individual $j \neq i$ is replaced by $\bar{R}$. That is, $s_i = (x_1, \ldots, x_n, \bar{R}, \ldots, \bar{R}, R_i, \bar{R}, \ldots, \bar{R})$. Second, let $s_{ij}$ denote the social state for the two-individual population $\{i, j\} \subseteq N$ in which the preference relation of individual $j$ is replaced by $\bar{R}$. That is, $s_{ij} = (x_i, x_j, R_i, \bar{R})$. In the social state $s_i$, the only envy that occurs is that of individual $i$ towards all other individuals in $N$. Likewise, in the social state $s_{ij}$, the only envy that occurs

**Table 1.** Five envy measures

| Measure | $E(s)$ | N | A | D | B | P | $r$-P |
|---|---|---|---|---|---|---|---|
| First Feldman-Kirman | $\lvert\{\{i,j\} \subseteq N : x_j P_i x_i\}\rvert$ | yes | yes | yes | no | no | no |
| Second Feldman-Kirman* | $\sum_{i\in N}\sum_{j\in N}[u_i(x_j) - u_i(x_i)]$ | no | yes | yes | yes | yes‡ | yes‡ |
| Third Feldman-Kirman* | $\sum_{i\in N}\sum_{j\in N}\max\{u_i(x_j) - u_i(x_i), 0\}$ | yes | yes | yes | yes | yes‡ | yes‡ |
| Chaudhuri† | $\sum_{i\in N}\sum_{j\in N}\max\left\{\frac{1}{\lambda_{ij}} - 1, 0\right\}$ | yes | yes | yes | no | yes | yes |
| Diamantaras-Thomson† | $\max_{i,j\in N}\left\{\frac{1}{\lambda_{ij}} - 1\right\}$ | no | yes | no | no | yes | yes |

Abbreviations: N(ormalization), A(nonymity), D(ecomposability), B(etweenness), P(roportionality), $r$-P(roportionality).
*The function $u_i : \mathbb{R}^m_+ \to \mathbb{R}$ is a utility representation of $R_i$.
†The real number $\lambda_{ij}$ is such that $x_i I_i \lambda_{ij} x_j$.
‡The axiom is satisfied only for specific utility representations.

is that of individual $i$ towards individual $j$. We may therefore interpret $E(s_i)$ as the envy of individual $i$ in social state $s$ and $E(s_{ij})$ as the envy of individual $i$ towards individual $j$ in social state $s$. We will refer to $E(s_i)$ as the *individual envy* of $i$ and to $E(s_{ij})$ as the *elementary envy* of $i$ to $j$. Note that $E(s_i)$ and $E(s_{ij})$ are well defined for each envy measure $E$ and each social state $s$ in $S$.

To put the analysis of the subsequent sections into perspective, we consider several envy measures that have been proposed in the literature. None of these measures has received axiomatic foundations. Each measure has been motivated instead by its immediate intuitive appeal. Table 1 presents the envy measures proposed by Feldman and Kirman (1974), Chaudhuri (1986) and Diamantaras and Thomson (1989).[4] The table also shows how the measures fare with respect to the axioms defined in this section and subsequent sections.

All five measures rely on elementary envy as a basic building block. For

[4]Our formulation of the Diamantaras-Thomson measure follows Arnsperger (1994, Definition 5.4).

the first four measures in Table 1, overall envy $E(s)$ equals the sum of all elementary envies $\sum_{i\in N}\sum_{j\in N} E(s_{ij})$. For the final measure in the table, overall envy $E(s)$ equals the maximum elementary envy $\max_{i,j\in N} E(s_{ij})$.[5] Let us focus now on how each of the measures defines elementary envy.

For the first Feldman-Kirman measure, which is a simple count of the instances of envy, the elementary envy $E(s_{ij})$ equals 1 if individual $i$ envies individual $j$ and 0 if not. The measure clearly neglects the intensity of elementary envy, contrary to the next four measures in the table.

The second and third Feldman-Kirman measures assume each individual $i$ has a utility representation $u_i$ with cardinal significance, and compute the intensity of elementary envy using utility differences. The elementary envy $E(s_{ij})$ equals $u_i(x_j) - u_i(x_i)$ for the second Feldman-Kirman measure, and the same value truncated at zero for the third Feldman-Kirman measure. Hence, the former measure takes into account 'negative' elementary envies, i.e., the extent to which individuals prefer their own bundles to those of others, whereas the latter measure does not.[6] A shortcoming of these two measures is the dependence on the arbitrary choice of a utility representation $u_i$ for each individual $i$.

The Chaudhuri and Diamantaras-Thomson measures do not rely on cardinal utility information. Instead, these measures focus on the fraction $\lambda_{ij}$ by which the bundle of $j$ has to be shrunk in order for $i$ to stop envying $j$. For the Diamantaras-Thomson measure, elementary envy $E(s_{ij})$ equals $(1/\lambda_{ij}) - 1$, where $\lambda_{ij}$ is such that $x_i I_i \lambda_{ij} x_j$. For the Chaudhuri measure, elementary envy is the same value truncated at zero. Again, the Diamantaras-Thomson measure takes into account 'negative' elementary envies, whereas the Chaudhuri measure does not. A shortcoming of these two measures is their arbitrary de-

[5]Hence, the five measures in Table 1 depend only indirectly on the individual envies. The individual envy of $i$ to all other individuals in $N$ equals $E(s_i) = \sum_{j\in N} E(s_{ij})$ for the first four measures in the table and $E(s_i) = \max_{j\in N} E(s_{ij})$ for the final measure. In Section 5 we discuss a measure by Fleurbaey (2008, Chapter 2) that gives a more substantial role to the individual envies.

[6]Feldman and Kirman (1974, p. 997) introduce their third measure with the explicit objective of measuring envy without taking into account 'negative' elementary envies. It may indeed be argued that such 'negative' elementary envies should be considered as irrelevant in equity evaluations. Our axioms also neutralize their role (see Section 3).

pendence on the particular procedure of shrinking the bundle of the envied. An a priori equally appealing procedure would be to focus on the factor by which the bundle of the envious has to be blown up in order for him to stop envying, but this procedure yields different results (see Section 4).

We proceed as follows. In Section 3 we characterize an envy measure that equates overall envy to the sum of all elementary envies. The measure takes the form of the first four measures in Table 1. But measures that equate overall envy to the maximum elementary envy, as the Diamantaras-Thomson measure, or to the minimum elementary envy may be obtained as limiting cases (see Section 5). In Section 4 we consider axioms that only impose properties on an envy measure for the two-individual case. Using these axioms we characterize a measure of elementary envy that combines the utility difference approach of the second and third Feldman-Kirman measures and the radial distance approach of the Chaudhuri and Diamantaras-Thomson measures. But at the same time it avoids the shortcomings of these approaches. Section 5 combines the results of the two preceding sections into a single class of envy measures and discusses its properties.

## 3 Envy as the sum of elementary envies

To define decomposability, imagine a partitioning of the population into two subgroups, e.g., on the basis of region, ethnicity or gender. Decomposability conveniently allows to write overall envy in the population as a sum of the envy within subgroups and the envy between subgroups.[7]

**Decomposability.** For each social state $(x_N, R_N)$ in $S$ and each partition $\{N_1, N_2\}$ of $N$ with non-empty $N_1$ and $N_2$, we have

$$\begin{aligned} E(x_N, R_N) \;&=\; E(x_{N_1}, R_{N_1}) \;+\; E(x_{N_2}, R_{N_2}) \\ &+ \sum_{i \in N_1} E(x_i, x_{N_2}, R_i, \bar{R}, \ldots, \bar{R}) \;+\; \sum_{i \in N_2} E(x_i, x_{N_1}, R_i, \bar{R}, \ldots, \bar{R}). \end{aligned} \tag{1}$$

[7]Similar decomposability requirements have been studied in the context of inequality measurement (e.g., Bourguignon, 1979, Cowell, 1980, and Shorrocks, 1980, 1984). The decomposability axiom used here most resembles that of Ebert (2010). While we, as is customary, state the axiom in terms of two subgroups, repeated application of equation (1) allows a decomposition in any number of subgroups.

The first two terms in equation (1) constitute the within subgroup component, the final two terms the between subgroup component. Within subgroup envy is the sum of the envy levels in the two subgroups. Between subgroup envy is the sum of the individual envy of each individual toward the other subgroup.

The following lemma says that if an envy measure satisfies normalization and decomposability, then it measures individual envy by the sum of the individual's elementary envies.

**Lemma 1.** *If $E$ satisfies normalization and decomposability, then, for each social state $(x_N, R_N)$ in $S$ and each individual $i$ in $N$, we have*

$$E(s_i) \;=\; \sum_{j \in N} E(x_i, x_j, R_i, \bar{R}).$$

*Proof.* The proof is by induction on the number of individuals.

*Step 1.* Let $N$ be such that $|N| = 2$. Without loss of generality, let $N = \{i_1, i_2\}$. Let $s = (x_{i_1}, x_{i_2}, R_{i_1}, R_{i_2})$ be a social state. By decomposability,

$$\begin{aligned} E(s_{i_1}) \;=\; & E(x_{i_1}, R_{i_1}) \;+\; E(x_{i_2}, \bar{R}) \\ & +\; E(x_{i_1}, x_{i_2}, R_{i_1}, \bar{R}) \;+\; E(x_{i_2}, x_{i_1}, \bar{R}, \bar{R}). \end{aligned}$$

The first, second and final terms are equal to zero by normalization. Hence,

$$E(s_{i_1}) \;=\; E(x_{i_1}, x_{i_2}, R_{i_1}, \bar{R}).$$

*Step 2.* Suppose the hypothesis holds for all $N$ such that $|N| = n$ with $n \geq 2$. We have to show that it holds for all social states with $n+1$ individuals.

Let $N'$ be such that $|N'| = n+1$ and let $s = (x_{N'}, R_{N'})$ be a social state. Without loss of generality, let $N' = \{i_1, i_2, \ldots, i_{n+1}\}$. Consider the partitioning of $N'$ into $N'_1 = \{i_1, i_2, \ldots, i_n\}$ and $N'_2 = \{i_{n+1}\}$. By decomposability,

$$\begin{aligned} E(s_{i_1}) \;=\; & E(x_{N'_1}, R_{i_1}, \bar{R}, \ldots, \bar{R}) \;+\; E(x_{i_{n+1}}, \bar{R}) \\ +\; & E(x_{i_1}, x_{i_{n+1}}, R_{i_1}, \bar{R}) \;+\; \sum_{k=2}^{n} E(x_{i_k}, x_{i_{n+1}}, \bar{R}, \bar{R}) \;+\; E(x_{i_{n+1}}, x_{N'_1}, \bar{R}, \ldots, \bar{R}). \end{aligned}$$

The second, fourth and final terms are equal to zero by normalization. Hence,

$$E(s_{i_1}) \;=\; E(x_{N'_1}, R_{i_1}, \bar{R}, \ldots, \bar{R}) \;+\; E(x_{i_1}, x_{i_{n+1}}, R_{i_1}, \bar{R}).$$

By the induction hypothesis,

$$E(x_{N'_1}, R_{i_1}, \bar{R}, \ldots, \bar{R}) \;=\; \sum_{k=1}^{n} E(x_{i_1}, x_{i_k}, R_{i_1}, \bar{R}).$$

Hence,

$$E(s_{i_1}) \;=\; \sum_{k=1}^{n+1} E(x_{i_1}, x_{i_k}, R_{i_1}, \bar{R}).$$

□

The following proposition says that an envy measure satisfies normalization and decomposability if and only if overall envy equals the sum of all elementary envies.

**Proposition 1.** *An envy measure $E$ satisfies normalization and decomposability if and only if, for each social state $(x_N, R_N)$ in $S$, we have*

$$E(x_N, R_N) \;=\; \sum_{i \in N} \sum_{j \in N} E(x_i, x_j, R_i, \bar{R}),$$

*where $E(x_i, x_j, R_i, \bar{R}) = 0$ for all individuals $i$ and $j$ in $N$ such that $x_i R_i x_j$.*

*Proof.* It is easy to see that the stated envy measure satisfies normalization and decomposability. We focus on the reverse implication. The proof is by induction on the number of individuals.

*Step 1.* Let $N$ be such that $|N| = 2$. Without loss of generality, let $N = \{i_1, i_2\}$. Let $s = (x_{i_1}, x_{i_2}, R_{i_1}, R_{i_2})$ be a social state. By decomposability,

$$\begin{aligned} E(s) \;&=\; E(x_{i_1}, R_{i_1}) \;+\; E(x_{i_2}, R_{i_2}) \\ &\quad+\; E(x_{i_1}, x_{i_2}, R_{i_1}, \bar{R}) \;+\; E(x_{i_2}, x_{i_1}, R_{i_2}, \bar{R}). \end{aligned}$$

The first and second terms are equal to zero by normalization. Hence,

$$E(s) \;=\; E(x_{i_1}, x_{i_2}, R_{i_1}, \bar{R}) \;+\; E(x_{i_2}, x_{i_1}, R_{i_2}, \bar{R}).$$

*Step 2.* Suppose the hypothesis holds for all $N$ such that $|N| = n$ with $n \geq 2$. We have to show that it holds for all social states with $n + 1$ individuals.

Let $N'$ be such that $|N'| = n + 1$ and let $s = (x_{N'}, R_{N'})$ be a social state. Without loss of generality, let $N' = \{i_1, i_2, \ldots, i_{n+1}\}$. Consider the partitioning of $N'$ into $N_1' = \{i_1, i_2, \ldots, i_n\}$ and $N_2' = \{i_{n+1}\}$. By decomposability,

$$\begin{aligned} E(s) \;&=\; E(x_{N_1'}, R_{N_1'}) \;+\; E(x_{i_{n+1}}, R_{i_{n+1}}) \\ &\quad + \sum_{k=1}^{n} E(x_{i_k}, x_{i_{n+1}}, R_{i_k}, \bar{R}) \;+\; E(x_{i_{n+1}}, x_{N_1'}, R_{i_{n+1}}, \bar{R}, \ldots, \bar{R}). \end{aligned}$$

The second term is equal to zero by normalization. Using, in addition, the induction hypothesis,

$$\begin{aligned} E(s) \;&=\; \sum_{k=1}^{n}\sum_{\ell=1}^{n} E(x_{i_k}, x_{i_\ell}, R_{i_k}, \bar{R}) \\ &\quad + \sum_{k=1}^{n} E(x_{i_k}, x_{i_{n+1}}, R_{i_k}, \bar{R}) \;+\; E(x_{i_{n+1}}, x_{N_1'}, R_{i_{n+1}}, \bar{R}, \ldots, \bar{R}). \end{aligned}$$

By Lemma 1,

$$E(x_{i_{n+1}}, x_{N_1'}, R_{i_{n+1}}, \bar{R}, \ldots, \bar{R}) \;=\; \sum_{k=1}^{n+1} E(x_{i_{n+1}}, x_{i_k}, R_{i_{n+1}}, \bar{R}).$$

Hence,

$$E(s) \;=\; \sum_{k=1}^{n+1}\sum_{\ell=1}^{n+1} E(x_{i_k}, x_{i_\ell}, R_{i_k}, \bar{R}).$$

□

Proposition 1 says that overall envy equals the sum of all elementary envies, but largely leaves open how to measure elementary envy. All it imposes in this respect is that 'negative' elementary envies, i.e., the extent to which individuals prefer their own bundles to those of others, are not taken into account. In the next section, we will consider axioms that give more content to the concept of elementary envy.

## 4 Measuring elementary envy

Consider a setting with two individuals, only one of whom is envious. We propose axioms that require the envy measure to react to simple changes in the bundles of the two individuals. The axioms bare on the envy measure $E$, but only directly impose properties on the elementary envy measure corresponding to $E$.

Betweenness requires the elementary envy of individual $i$ to individual $j$ to decrease if $i$'s bundle improves or $j$'s bundle worsens according to $i$'s preferences. In terms of $i$'s preferences, the new bundles lie 'in between' the original bundles.

**Betweenness.** For all individuals $i$ and $j$ in $\mathcal{N}$, all bundles $x_i$, $x_j$, $x_i'$ and $x_j'$ in $X$ and each preference relation $R_i$ in $\mathcal{R}$ such that $x_j P_i x_i$, we have that $x_j R_i x_j'$, $x_j' R_i x_i'$ and $x_i' R_i x_i$ imply $E(x_i, x_j, R_i, \bar{R}) \geq E(x_i', x_j', R_i, \bar{R})$ with strict inequality holding whenever $x_j P_i x_j'$ or $x_i' P_i x_i$.

We emphasize an implication of betweenness. Let $u_i$ be a utility representation of the preference relation $R_i$. Betweenness implies that the elementary envy $E(x_i, x_j, R_i, \bar{R})$ of individual $i$ to $j$ is a function only of the utility levels $u_i(x_i)$ and $u_i(x_j)$. That is, if $u_i(x_i) = u_i(x_i')$ and $u_i(x_j) = u_i(x_j')$ (as depicted in Figure 1), then $E(x_i, x_j, R_i, \bar{R}) = E(x_i', x_j', R_i, \bar{R})$. Note that the second and third Feldman-Kirman measures are in this functional form and satisfy betweenness.

The next axiom captures the idea of gauging elementary envy by the radial distance between bundles. Consider two approaches. The first approach, as adopted in the Chaudhuri and Diamantaras-Thomson measures, measures the elementary envy of $i$ to $j$ using the fraction $\lambda_{ij}$ by which $j$'s bundle has to be shrunk in order for $i$ to stop envying $j$. That is, $\lambda_{ij}$ is such that $x_i I_i \lambda_{ij} x_j$, and the higher $\lambda_{ij}$, the lower the elementary envy of $i$ to $j$. The second approach measures the elementary envy of $i$ to $j$ using the factor $\kappa_{ij}$ by which $i$'s bundle has to be blown up in order for $i$ to stop envying $j$. That is, $\kappa_{ij}$ is such that $\kappa_{ij} x_i I_i x_j$, and the higher $\kappa_{ij}$, the higher the elementary envy of $i$ to $j$. The two approaches are a priori equally appealing, but yield conflicting results. To see this, consider the social states $s^1 = (x_i, x_j', R_i, \bar{R})$ and $s^2 = (x_i', x_j, R_i, \bar{R})$ in Figure 1. The first approach implies $E(s^1) > E(s^2)$ because $\lambda_{ij}' < \lambda_{ij}$, whereas the second approach implies $E(s^1) < E(s^2)$ because $\kappa_{ij} < \kappa_{ij}'$.[8]

[8]The two approaches do give the same result if the preference relation is homothetic.

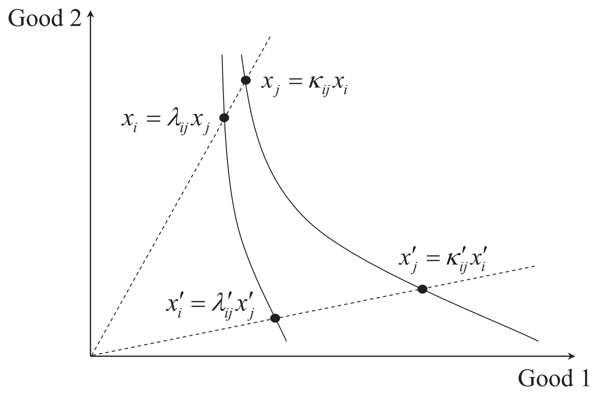


*Figure 1.* Two indifference curves of individual $i$

We do not make a choice among the two conflicting approaches. Instead, we formulate an axiom that is sufficiently weak to be consistent with both. The axiom only considers the cases where the bundles of the envied and envious are proportional to each other (in which case the two above approaches coincide) and says that a decrease of the radial distance between these two bundles reduces elementary envy.

**Proportionality.** For all individuals $i$ and $j$ in $\mathcal{N}$, all bundles $x_i$, $x_j$, $x'_i$ and $x'_j$ in $X$ such that $\kappa x_i = x_j$ and $\kappa' x'_i = x'_j$ and all preference relations $R_i$ and $R'_i$ in $\mathcal{R}$ such that $x_j P_i x_i$ and $x'_j P'_i x'_i$, we have that $\kappa \geq \kappa'$ implies $E(x_i, x_j, R_i, \bar{R}) \geq E(x'_i, x'_j, R'_i, \bar{R})$ with strict inequality holding if and only if $\kappa > \kappa'$.

However, betweenness and proportionality are incompatible: there is no envy measure that satisfies both axioms. Consider the social states $s = (x_i, x_j, R_i, \bar{R})$ and $s' = (x'_i, x'_j, R_i, \bar{R})$ in Figure 1. Betweenness implies $E(s) = E(s')$, whereas proportionality implies $E(s) < E(s')$ because $\kappa_{ij} < \kappa'_{ij}$.[9] Note that a stronger clash, with betweenness implying $E(s) > E(s')$, can easily be constructed as well.

We treat betweenness as essential and therefore weaken proportionality. The following axiom requires all bundles to be proportional to a predeter-

[9]If the domain of preference relations is restricted to homothetic preferences relations, then the two axioms are compatible.

mined reference bundle $r$. Later we will argue that the axiom may be regarded as a minimal weakening of proportionality that is compatible with betweenness.

***r*-Proportionality.** There is a bundle $r$ in $X$ such that the following holds. For all individuals $i$ and $j$ in $\mathcal{N}$, all bundles $x_i$, $x_j$, $x'_i$ and $x'_j$ in $X$ proportional to $r$ and such that $\kappa x_i = x_j$ and $\kappa' x'_i = x'_j$ and all preference relations $R_i$ and $R'_i$ in $\mathcal{R}$ such that $x_j P_i x_i$ and $x'_j P'_i x'_i$, we have that $\kappa \geq \kappa'$ implies $E(x_i, x_j, R_i, \bar{R}) \geq E(x'_i, x'_j, R'_i, \bar{R})$ with strict inequality holding if and only if $\kappa > \kappa'$.

Before proceeding, we need to define the $\rho$-utility representation. Let $\rho$ be a reference bundle in $X$. Let $u_\rho(x_i, R_i)$ be the real number such that individual $i$ is indifferent between the fraction $u_\rho(x_i, R_i)$ of the bundle $\rho$ and his own bundle $x_i$. That is, for a preference relation $R_i$ in $\mathcal{R} \setminus \{\bar{R}\}$, we have that $u_\rho(x_i, R_i)$ is the real number such that $x_i I_i u_\rho(x_i, R_i)\rho$. For the preference relation $R_i = \bar{R}$, we let $u_\rho(x_i, R_i)$ equal a positive constant. The function $u_\rho(\,\cdot\,, R_i)$ is a utility representation of the preference relation $R_i$.[10]

The following proposition says that an envy measure satisfies betweenness and $r$-proportionality if and only if it measures the elementary envy of individual $i$ to $j$ by the ratio of $i$'s $\rho$-utility levels associated with $j$'s and $i$'s bundles. Moreover, the reference bundle $\rho$ that determines the utility representation has to be chosen such that $\rho = r$.

**Proposition 2.** *Let $E$ be an envy measure that satisfies anonymity. Then $E$ satisfies betweenness and $r$-proportionality if and only if there exists a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that for all individuals $i$ and $j$ in $\mathcal{N}$, all bundles $x_i$ and $x_j$ in $X$ and each preference relation $R_i$ in $\mathcal{R}$ such that $x_j P_i x_i$, we have*

$$E(x_i, x_j, R_i, \bar{R}) \;=\; f\left(\frac{u_r(x_j, R_i)}{u_r(x_i, R_i)}\right).$$

*Proof.* It is easy to see that the stated envy measure satisfies betweenness and $r$-proportionality. We focus on the reverse implication.

[10] See Fleurbaey and Maniquet (2011, p. 7) for a discussion of the $\rho$-utility representation.

Let $r$ be a given bundle in $X$. Let $i$ and $j$ be individuals in $\mathcal{N}$, let $x_i$, $x'_i$, $x_j$ and $x'_j$ be bundles in $X$ and let $R_i$ and $R'_i$ be preference relations in $\mathcal{R}$ such that $x_j P_i x_i$ and $x'_j P'_i x'_i$. We have to show that

$$E(x_i, x_j, R_i, \bar{R}) \;\geq\; E(x'_i, x'_j, R'_i, \bar{R}) \tag{2}$$

if and only if

$$\frac{u_r(x_j, R_i)}{u_r(x_i, R_i)} \;\geq\; \frac{u_r(x'_j, R'_i)}{u_r(x'_i, R'_i)}. \tag{3}$$

Then there exists a strictly increasing function $f$ as stated. Note that $f$ does not depend on $i$ and $j$ by anonymity.

Let $y_i$, $y'_i$, $y_j$ and $y'_j$ be bundles in $X$ proportional to $r$ and such that $y_i I_i x_i$, $y'_i I'_i x'_i$, $y_j I_i x_j$ and $y'_j I'_i x'_j$. Such bundles exist since $R_i$ and $R'_i$ are continuous and strictly monotonic. Let $\kappa$ and $\kappa'$ be such that $\kappa y_i = y_j$ and $\kappa' y'_i = y'_j$.

Suppose that equation (2) holds. We have to show that equation (3) holds as well. By betweenness, we have $E(x_i, x_j, R_i, \bar{R}) = E(y_i, y_j, R_i, \bar{R})$ and $E(x'_i, x'_j, R'_i, \bar{R}) = E(y'_i, y'_j, R'_i, \bar{R})$. Hence, we obtain $E(y_i, y_j, R_i, \bar{R}) \geq E(y'_i, y'_j, R'_i, \bar{R})$. If $\kappa < \kappa'$, then $E(y_i, y_j, R_i, \bar{R}) < E(y'_i, y'_j, R'_i, \bar{R})$ by $r$-proportionality. Hence, it must be that $\kappa \geq \kappa'$. From the definition of $u_r$, it follows that $\kappa = u_r(y_j, R_i)/u_r(y_i, R_i)$ and $\kappa' = u_r(y'_j, R'_i)/u_r(y'_i, R'_i)$. Since $u_r(x_j, R_i)/u_r(x_i, R_i) = u_r(y_j, R_i)/u_r(y_i, R_i)$ and $u_r(x'_j, R'_i)/u_r(x'_i, R'_i) = u_r(y'_j, R'_i)/u_r(y'_i, R'_i)$, we obtain equation (3).

Now, suppose that equation (3) holds. We have to show that equation (2) holds as well. Equation (3) implies that $\kappa = u_r(y_j, R_i)/u_r(y_i, R_i) \geq \kappa' = u_r(y'_j, R'_i)/u_r(y'_i, R'_i)$. Since $\kappa \geq \kappa'$, we have $E(y_i, y_j, R_i, \bar{R}) \geq E(y'_i, y'_j, R'_i, \bar{R})$ by $r$-proportionality. Using betweenness, we obtain equation (2). $\square$

The measure of elementary envy in Proposition 2 shares with the second and third Feldman-Kirman measures that it depends on the utility distance between the bundles of the envious and the envied. However, the utility representation used is not an arbitrary choice as in those measures. Rather, the $\rho$-utility representation is singled out by the radial distance idea inherent in the Chaudhuri and Diamantaras-Thomson measures.

Note that, for a given individual $i$, the criterion in Proposition 2 provides a complete ranking of all social states of the form $(x_i, x_j, R_i, \bar{R})$. This means that any further strengthening of $r$-proportionality in the direction of proportionality will either lead to conflicts or is already implied by the combination of $r$-proportionality and betweenness. In this sense, $r$-proportionality is the minimal weakening of proportionality that is compatible with betweenness.

## 5 Main result and discussion

Our main result characterizes the class of envy measures satisfying normalization, anonymity, decomposability, betweenness and $r$-proportionality. The theorem is a straightforward combination of Propositions 1 and 2.

**Theorem 1.** *An envy measure $E$ satisfies normalization, anonymity, decomposability, betweenness and $r$-proportionality if and only if there exists a function $f : \mathbb{R}_{++} \to \mathbb{R}_+$ with $f$ strictly increasing on the interval $(1, +\infty)$ and $f(t) = 0$ for each $t \leq 1$ such that, for each social state $(x_N, R_N)$ in $S$, we have*

$$E(x_N, R_N) = \sum_{i \in N} \sum_{j \in N} f\left(\frac{u_r(x_j, R_i)}{u_r(x_i, R_i)}\right). \tag{4}$$

To understand the role of the function $f$, it is useful to treat the utility ratio $u_r(x_j, R_i)/u_r(x_i, R_i)$ as a natural cardinalization of the measure of elementary envy of individual $i$ to individual $j$. The more convex is $f$, the more sensitive is the envy measure $E$ to changes in larger elementary envies (as measured by the utility ratio) relative to changes in smaller elementary envies. Given a sufficiently convex $f$, the measure that equates overall envy to the largest elementary envy, as in the Diamantaras-Thomson measure, can be approximated arbitrarily closely. Similarly, choosing $f$ sufficiently concave delivers the other extreme that identifies overall envy with the minimal elementary envy.

We discuss two variants of Theorem 1. The first concerns the aggregation of elementary envies into overall envy, the second the definition of elementary envy.

First, not all envy measures that have been proposed take the form of a sum over the elementary envies. Fleurbaey's (2008, Chapter 2) measure equates the individual envy of $i$ to his maximal elementary envy $E(s_i) = \max_{j \in N} E(s_{ij})$ and overall envy to the sum of all individual envies $E(s) =$

$\sum_{i\in N} E(s_i)$.[11] This measure, contrary to the measures in Theorem 1, does not depend only on the values of the elementary envies, but also on their distribution over the individuals. Such a genuine role for the individual envies can be allowed by replacing decomposability by two simple positive responsiveness axioms. The first axiom requires individual envy to increase if at least one individual's elementary envy increases, other things equal. The second axiom requires overall envy to increase if at least one individual envy increases, other things equal. These axioms lead to a general approach that allows different aggregations for the elementary envies into individual envy and for the individual envies into overall envy. We omit the straightforward formal treatment.

Second, we examine how the measure of elementary envy changes if we focus on the absolute distance between bundles instead of on the relative distance. The only change to the assumptions in Section 2 is that commodities can take negative or zero values in addition to positive values. We use $1_m$ to denote the $m$-vector with a one at each entry. Consider the following absolute version of $r$-proportionality. Let $x_i$, $x_j$, $x'_i$ and $x'_j$ be bundles that are translations of the reference bundle $r$, and $x_i + \mu 1_m = x_j$ and $x'_i + \mu' 1_m = x'_j$. Let $x_j P_i x_i$ and $x'_j P_i x'_i$. Then, according to the alternative axiom, $\mu \geq \mu'$ implies $E(x_i, x_j, R_i, \bar{R}) \geq E(x'_i, x'_j, R'_i, \bar{R})$ with strict inequality holding if and only if $\mu > \mu'$. Replacing $r$-proportionality by this alternative axiom in Theorem 1 yields the following class of measures: for each social state $(x_N, R_N)$ in $S$, we have

$$E(x_N, R_N) \; = \; \sum_{i\in N}\sum_{j\in N} g\left(v_r(x_j, R_i) - v_r(x_i, R_i)\right), \tag{5}$$

where $v_r(x_i, R_i)$ is the real number such that $x_i I_i (r + v_r(x_i, R_i)1_m)$, and $g : \mathbb{R} \to \mathbb{R}$ is a function with $g$ strictly increasing on the interval $(0, +\infty)$ and $g(t) = 0$ for each $t \leq 0$. The proof involves a simple adaptation of the proof of Proposition 2 and is therefore omitted.

[11] Fleurbaey's approach to measure elementary envy is similar to that used in the Chaudhuri and Diamantaras-Thomson measures.

## 6 Concluding remark

We conclude with a remark on the connection between envy measures and inequality measures. The literature on multidimensional inequality measurement focuses on the setting with multiple commodities but homogenous preferences.[12] If $R_i = R$ for each individual $i$, then the envy measures in equations (4) and (5) correspond to so-called two-stage inequality measures.[13] The first stage computes the utility vectors, $(u_r(x_i, R))_{i\in N}$ for (4) and $(v_r(x_i, R))_{i\in N}$ for (5), and the second stage applies a unidimensional (income) inequality measure to these utility vectors. For the second stage, it is easy to obtain well-known unidimensional inequality measures such as the absolute Gini index, the variance and the variance of logarithms as special cases of equations (4) or (5).[14] This connection suggests envy measurement as a generalization of the two-stage approach to the setting of heterogenous preferences.

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[12] See Weymark (2006) and Chakravarty (2009) for surveys. While contributions to this literature do not always explicitly define preferences, an interpretation in terms of (homogenous) preferences is usually valid (see e.g., Tsui, 1995, pp. 252-253).

[13] Maasoumi (1986) introduced the two-stage approach. See Dardanoni (1995), Weymark (2006) and Bosmans, Decancq and Ooghe (2013) for further discussions.

[14] See Ebert (2010) for definitions of these and other measures covered by equations (4) and (5). In a fixed-population setting, the envy measures and inequality measures correspond exactly. In a variable-population setting, division by $2|N|^2$ is required to obtain the inequality measures.

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